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SOLUTIONS.

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R. J. Adcock 265; S. M. Barton 263, 267, 268; J. H. Boyd 263, 268; G. C. Comstock 263, 268, 269; Artemas Martin 246, 247, 263, 265, 266; T. U. Taylor 263, 268, 269; Wm. M. Thornton 264; W. O. Whitescarver 267, 268; L. G. Weld, 210, 263, 267, 268, 269, 270.

263

INTEGRATE $\frac{(x-1)\,xdx}{x^3+x^2+x+1}$, from $x=0$ to $x=1$. [*O. Root, Jr.*]

SOLUTION.

$$\int_1^0 \frac{(x-1)\,xdx}{x^3+x^2+x+1} = \int_1^0 \frac{dx}{x+1} - \int_1^0 \frac{1}{x^2+1}$$

$$= \left[\log(x+1) - \tan^{-1}x\right]_1^0$$

$$= \frac{1}{4}\pi - \log 2.$$

[*T. U. Taylor.*]

264

INTEGRATE $\frac{d\varphi}{\sqrt{(1-3\tan^2\varphi)}}$, from $\varphi=0$ to $\varphi=\frac{1}{4}\pi$. [*O. Root, Jr.*]

SOLUTION.

The exercise, if it has any significance, must mean

$$\int_0^{\frac{1}{6}\pi} \frac{d\varphi}{\sqrt{1-3\tan^2\varphi}} + i\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \frac{d\varphi}{\sqrt{3\tan^2\varphi-1}};$$

or, if $x = 2\sin\varphi$,

$$\tfrac{1}{2}\int_0^1 \frac{dx}{\sqrt{1-x^2}} + \tfrac{1}{2}i\int_1^{\sqrt{2}} \frac{dx}{\sqrt{x^2-1}}.$$

This equals

$$\tfrac{1}{2}\sin^{-1}x\Big]_0^1 + \tfrac{1}{2}i\log\left(x+\sqrt{x^2-1}\right)\Big]_1^{\sqrt{2}},$$

which equals

$$\frac{1}{4}\pi + \tfrac{1}{2}i\log(1+\sqrt{2}).$$

[*Wm. M. Thornton.*]

265

INTEGRATE the equation

$$dx = \sqrt{[a - \sqrt{(b^2 + x^2)}]}dy. \qquad [\textit{Artemas Martin.}]$$

SOLUTION.

Let

$$z^2 = a - \sqrt{(b^2 + x^2)} = (a - b)\sin^2 A\,;$$

then

$$dy = -\frac{2(a - z^2)dz}{\sqrt{[(a-z^2)^2 - b^2]}} = -\frac{2(a - z^2)dz}{\sqrt{(a + b - z^2)}\,.\,\sqrt{(a - b - z^2)}}$$

$$= -- \frac{2[a - (a-b)\sin^2 A]dA}{\sqrt{[a + b - (a-b)\sin^2 A]}} = -\frac{2a}{\sqrt{(a+b)}}\,\frac{dA}{\sqrt{\left[1 - \frac{a-b}{a+b}\sin^2 A\right]}}$$

$$+\frac{2(a-b)}{\sqrt{(a+b)}}\,\frac{\sin^2 A\,dA}{\sqrt{\left[1 - \frac{a-b}{a+b}\sin^2 A\right]}}$$

Since

$$\frac{\sin^2 A\,dA}{\sqrt{(1 - e^2\sin^2 A)}} = \frac{1}{e^2}\left[\frac{dA}{\sqrt{(1 - e^2\sin^2 A)}} - \sqrt{(1 - e^2\sin^2 A)}\,.\,dA\right],$$

$$y = \left[-\frac{2a}{\sqrt{(a+b)}} + 2\sqrt{(a+b)}\right]\int\frac{dA}{\sqrt{\left[1 - \frac{a-b}{a+b}\sin^2 A\right]}}$$

$$- 2\sqrt{(a+b)}\int\left[1 - \frac{a-b}{a+b}\sin^2 A\right]dA$$

$$= \frac{2b}{\sqrt{(a+b)}}F(e, A) - 2\sqrt{(a+b)}\,.\,E(e, A);$$

where

$$e = \sqrt{\frac{a-b}{a+b}},\quad A = \sin^{-1}\sqrt{\frac{a - \sqrt{(b^2 + x^2)}}{a - b}}.$$

[*R. J. Adcock.*]

266

It is required to find nine biquadrate numbers whose sum is a square number. [*Artemas Martin.*]

SOLUTION.

Let *ad*, *ae*, *af*, *bd*, *be*, *bf*, *cd*, *ce*, and *cf* represent the *roots* of the nine biquadrates sought; then must

$$a^4d^4 + a^4e^4 + a^4f^4 + b^4d^4 + b^4e^4 + b^4f^4 + c^4d^4 + c^4e^4 + c^4f^4 = (a^4 + b^4 + c^4)(d^4 + e^4 + f^4) = \square.$$

This will be the case if we make

$$a^4 + b^4 + c^4 = \square, \quad \text{and} \quad d^4 + e^4 + f^4 = \square.$$

Put $a^4 + b^4 + c^4 = (a^2 + b^2 - c^2)^2 = a^4 + 2a^2b^2 + b^4 - 2a^2c^2 - 2b^2c^2 + c^4$,

and we get $$a^2b^2 = a^2c^2 + b^2c^2, = c^2(a^2 + b^2) = \square,$$

which will have place if

$$a^2 + b^2 = \square.$$

Assume $$a = m(p^2 - q^2) \quad \text{and} \quad b = 2mpq,$$

then $$ab = cm(p^2 + q^2), = 2m^2pq(p^2 - q^2);$$

whence $$c = \frac{2m(p^2 - q^2)}{p^2 + q^2}.$$

Now take $m = p^2 + q^2$ and we have

$$a = p^4 - q^4, \quad b = 2pq(p^2 + q^2), \quad c = 2pq(p^2 - q^2).$$

Putting $$d^4 + e^4 + f^4 = (d^2 + e^2 - f^2)^2,$$

we find in the same way

$$d = r^4 - s^4, \quad e = 2rs(r^2 + s^2), \quad f = 2rs(r^2 - s^2).$$

Substituting these values for a, b, c, d, e, f, we have

$$(a^4 + b^4 + c^4)(d^4 + e^4 + f^4) = (p^8 + 14p^4q^4 + q^4)^2 (r^8 + 14r^4s^4 + s^8)^2.$$

Therefore, the general values of the *roots* of the required numbers are

$$\begin{aligned} ad &= (p^4 - q^4)(r^4 - s^4), & ae &= 2rs(r^2 + s^2)(p^4 - q^4), \\ af &= 2rs(r^2 - s^2)(p^4 - q^4), & bd &= 2pq(p^2 + q^2)(r^4 - s^4), \\ be &= 4pqrs(p^2 + q^2)(r^2 + s^2), & bf &= 4pqrs(p^2 + q^2)(r^2 - s^2), \\ cd &= 2pq(p^2 - q^2)(r^4 - s^4), & ce &= 4pqrs(p^2 - q^2))r^2 + s^2), \end{aligned}$$
$$cf = 4\,pqrs(p^2 - q^2)(r^2 - s^2);$$

where p, q, r, s may be any numbers.

Taking $p = 2$, $q = 1$, $r = 3$, $s = 2$, we find $a = 15$, $b = 20$, $c = 12$, $d = 65$, $e = 156$, $f = 60$ and the required numbers are

$$720^4,\ 780^4,\ 900^4,\ 975^4,\ 1200^4,\ 1300^4,\ 1872^4,\ 2340^4,\ 3120^4,$$

whose sum is 12006241^2. [*Artemas Martin.*]

267

EXAMINE for asymptotes $\rho^2 = a^2 \dfrac{\sin 3\theta}{\cos \theta}$. [*O. Root, Jr.*]

SOLUTION.

Differentiating and reducing

$$\frac{d\rho}{d\theta} = \tfrac{1}{2}\rho\,(3 \cot 3\theta + \tan \theta),$$

or

$$\rho^2 \frac{d\theta}{d\rho} = \frac{2\rho}{3 \cot 3\theta + \tan \theta}$$

$$= 2a \frac{\sin^{\frac{3}{2}} 3\theta \cos^{\frac{1}{2}}\theta}{3 \cos 3\theta \cos \theta + \sin 3\theta \sin \theta}.$$

For $\theta = \frac{1}{2}\pi$ and $\frac{3}{2}\pi$,

$$\rho = \infty \quad \text{and} \quad \rho^2 \frac{d\theta}{d\rho} = 0;$$

hence, for these values of θ, the radius vector is an asymptote.

It is easily shown that ρ has maximum values for $\theta = \frac{1}{2}\pi$, $\frac{3}{2}\pi$, and $\cos^{-1} \pm \frac{1}{2} \sqrt{(1 \pm \sqrt{3})}$. The curve is of the fifth order and is imaginary between $\theta = 60°$ and $90°$, $120°$ and $180°$, $240°$ and $270°$, and $300°$ and $360°$. The origin is a triple point. One of the corresponding branches is serpentine and lies in the second and fourth quadrants; the other two unite to form a figure 8 the loops of which lie in the first and third quadrants.

[*S. M. Barton.*]

268

FIND the maximum rectangle inscribed in the witch $y^2 + a^2 = \frac{a^3}{x}$.

[*O. Root, Jr.*]

SOLUTION.

The area of the inscribed rectangle is

$$2xy = 2a\,[(a - x)x]^{\frac{1}{2}}.$$

Here x and $(a - x)$ are two parts into which a is divided; hence their product is a maximum when they are equal, that is, when $x = \frac{1}{2}a$. Hence a^2 is the maximum area sought. [*L. G. Weld.*]

This example is found in Taylor's Calculus, p. 268.

269

SUPPOSE a series of ellipses described on the same conjugate axis. Find the eccentricity of that one for which the distance from the centre to the directrix is a minimum. [*H. B. Newson.*]

SOLUTION.

The equations of the ellipses are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where b is constant, and a is a variable parameter. The distance from the centre of an ellipse to its directrix being $\frac{a}{e} = \frac{b}{e\sqrt{1 - e^2}}$, we are to make this latter expression a minimum. Put $e = \sin\varphi$, and the expression becomes $\frac{2b}{\sin 2\varphi}$, which is obviously a minimum when $\varphi = \frac{1}{4}\pi$, hence the required eccentricity is $e = \sin\frac{1}{4}\pi = \frac{1}{2}\sqrt{2}$, and the minimum distance from centre to directrix is $2b$.

[*Geo. C. Comstock.*]

270

GIVEN a series of ellipses described on the same transverse axis. Find the eccentricity of that one for which the first pedal curve passes from a smooth oval form to an indented form having four points of inflection.

[*H. B. Newson.*]

SOLUTION.

We have for determining the equation of the pedal

$$r = x\cos\theta + y\sin\theta,$$

in which x and y refer to the given curve, and r and θ to the pedal. Also

$$\theta = \varphi - \tfrac{1}{2}\pi,$$

in which φ is the angle made by the tangent to the given curve with XX'.

From these, with the equation of the ellipse, we readily deduce for the equation of the pedal

$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta.$$

For the points common to the pedal and the tangent at the extremity of the axis minor we have the condition

$$r = b\operatorname{cosec}\theta,$$

which may be put in the form

$$\sin^4\theta + \frac{a^2}{b^2 - a^2}\sin^2\theta = \frac{b^2}{b^2 - a^2}.$$

The four roots of this equation in $\sin\theta$ must be equal, which condition gives

$$a^2 = 2b^2,$$

whence

$$e = \tfrac{1}{2}\sqrt{2}.$$

[*L. G. Weld.*]